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# Convection from a small, continuous source of heat in a calm, neutral atmosphere (with an appendix on the bent-over plume)

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## Summary

The problem of the free convection jet from a continuous source of heat in a neutral atmosphere at rest is discussed and formulae are derived which give the variation of the vertical velocity, w, the potential temperature excess, $\theta'$, and the radius of the jet with height, z. It is believed that the formulae may have a useful application in such problems as the rising plume of heated gas emitted from factory chimneys.

The solutions to the basic equations incorporate the usual power law features but one set, in particular, contains an exponential term which leads ultimately to a fairly rapid decay of vertical velocity with height. This should provide a more realistic picture in application to practical problems than the much slower rate of decay indicated by the simple power laws generally used.

Comparison of the theory with data from laboratory experiments suggests that, for small scale phenomena, the effects of vertical shearing stress may be neglected. The solutions, thus modified, give results in close agreement with laboratory data and include, as special cases, a number of formulae derived by dimensional and other arguments.

## 1. General discussion

We consider the velocity and temperature fields and also the vertical variation of radius in the ascending plume or jet of air which results when heat is supplied continuously from a point or a small circular source. In its most general terms the problem is intractable and can only be discussed in a descriptive manner. The difficulty consists largely of accounting for both thermal and dynamic instability, their causes and effects, in a rising air column which is surrounded by an atmosphere in motion and possessing its own thermal characteristics.

It is in consequence necessary to impose some limitations on the type of problem to be studied and, following the normal practice, we postulate that the atmosphere is at rest, except near the boundary of the rising air where turbulent mixing occurs, and in neutral equilibrium. The heated air may then be considered to rise vertically and to be symmetrical about a vertical axis in regard to lateral extent, velocity and temperature excess. At any level above the source, the upward velocity and the temperature excess have their maximum values on the axis of the jet and become smaller and smaller as the horizontal distance from the axis increases.

It is further assumed that while the temperature differences between the jet and the surrounding air are sufficient to produce vertical motion, the associated differences in density are small enough, in comparison with the

/density

density itself, to be ignored. Observational evidence supports this assumption since it is well known that a density variation of a few per cent may produce important vertical currents while the inertia of the fluid remains practically unaffected.

The problem is thus formulated as one of free convection and we apply the theories developed in fluid dynamics in relation to boundary layers, jets and wakes (see, for example, Goldstein: Modern Developments in Fluid Dynamics). This approach has also been used by Rouse, Yih and Humphreys (1952) and by Priestley and Ball (1955). These papers contain a discussion of the basic differential equations in their meteorological context.

The following notation is used:-

$z$ = vertical distance above source
$r$ = horizontal distance from z-axis
$R$ = lateral extent of jet = $R(z)$
$w$ = vertical velocity along z-axis = $w(z)$
$\theta_e$ = environment potential temperature (assumed constant)
$\theta$ = potential temperature
$\theta'$ = excess potential temperature along z-axis, $(\theta - \theta_e) = \theta'(z)$
$\tau$ = vertical shearing stress

At points removed from the z-axis the velocity and temperature excess are given by:-

$$w(z, r) = w(z)\, f(r/R)$$

$$\theta'(z, r) = \theta'(z)\, h(r/R)$$

These expressions are justified by considerations of dynamical similarity which suggest that the lateral variation of w has the same form at all heights and similarly for the lateral variation of $\theta'$.

It is further assumed that the vertical shearing stress $\tau$ is proportional to the square of the velocity, thus

$$\tau = \tfrac{1}{2} \rho w^2 j(r/R)$$

The functions f, h and j of $(r/R)$ are unknown but for most of what follows it is sufficient that their integrals, separately or in combination or in powers, should be finite.

As Rouse et al (1952) and Priestley and Ball (1955) have shown, the equations for momentum and of energy integrated over a horizontal plane from $r = 0$ to $r = \infty$, give the following differential equations governing R, the radius of the jet, and w and $\theta'$, the axial values of the vertical velocity and the temperature excess:-

$$\frac{d}{dz}(R^2 w^2) = a R^2 \frac{\theta'}{\theta_e} g \qquad (1)$$

$$\frac{d}{dz}(R^2 w^3) = b R^2 w \frac{\theta'}{\theta_e} g - c R w^3 \qquad (2)$$

In these equations a, b and c are numbers which could be derived directly if the functions f, h and j were known.

Another equation follows from our assumption that the atmosphere has constant potential temperatures. If the heat output Q (calories per second) at the source is constant, then the flow of heat across any horizontal cross-section must also be a constant. We may therefore write

$$R^2 w \theta' = \text{constant, } A \qquad (3)$$

/Priestley

Priestley and Ball (1955) have shown that equation (3) holds only if the atmosphere is in neutral equilibrium. Using this equation, we may write equations (1) and (2) as follows:-

$$w \frac{d}{dz} (R^2 w^2) = a \frac{Ag}{\theta_e} \tag{4}$$

$$\frac{d}{dz} (R^2 w^3) = b \frac{Ag}{\theta_e} - c\, Rw^3 \tag{5}$$

From these two equations $\frac{Ag}{\theta_e}$ can be eliminated and a single differential equation is obtained:-

$$(3a - 2b) \frac{1}{w} \frac{dw}{dz} + 2(a - b) \frac{1}{R} \frac{dR}{dz} = - \frac{ac}{R} \tag{6}$$

The solution of equation (6) and its significance will depend largely on the relative values of a and b. A fairly wide range of possibilities may be considered but in what follows attention is mainly confined to those considered likely to have an application to practical problems. Equation (6) could have been derived directly from equations (1) and (2) without the aid of equation (3), and in the following two sections equation (3) is not invoked although it does hold for some of the examples discussed.

## 2. Radius of the jet increasing linearly with height

In several papers on the free convection jet, linear spreading has been deduced at an early stage in the development or assumed. This law for the variation of R can be conveniently discussed as follows:-

(i) $3a - 2b = 0$

If $3a = 2b$, equation (6) reduces simply to

$$R = cz + \text{constant} \tag{7}$$

This is the solution found by Priestley and Ball (1955) for a neutral atmosphere and their solutions for w and $\theta'$ then follow from equations (3) and either (4) or (5).

The condition that $3a - 2b = 0$ requires that the functions $f(r/R)$ and $h(r/R)$ should be the same, that is, both w and $\theta'$ should have identical laws of lateral decay. It appears unlikely that this condition would hold in practice and, in the absence of experimental support for it, a certain degree of scepticism can hardly be avoided. For example, if this condition were to be fulfilled, the value of w for large values of z would be virtually proportional to $z^{-\frac{1}{3}}$ yielding velocities far larger than experience would lead us to expect.

(ii) $3a - 2b \neq 0$; $R \propto z$

Let $R = \epsilon z$

where $\epsilon$ is a constant. Substitution in equation (6) and integration lead to the result

$$w = w_o \left(\frac{z}{z_o}\right)^{-\frac{2(a-b) + \frac{ac}{\epsilon}}{3a - 2b}} \tag{8}$$

In the special case of c and $\epsilon$ being equal, as in the solution examined in (i) above, the expression for w would reduce to

$$w = w_o \left(\frac{z}{z_o}\right)^{-1} \tag{9}$$

and we should also have

$$\theta' = 0 \tag{10}$$



/On

On practical grounds the simultaneous solutions for w and $\theta'$ given by equations (9) and (10) are considered improbable.

Excluding the case when c and $\varepsilon$ are equal, there remains the possibility of a variety of solutions depending on the ratio $c/\varepsilon$ as well as on the ratio a/b. Because of the assumption of linear spreading of the jet, all these solutions for w, and hence $\theta'$, are power law solutions.

If a = b, the expressions for w and $\theta'$

are

$$w = w_o \left(\frac{z}{z_o}\right)^{-c/\varepsilon}$$

$$\theta' \propto z^{-(2 - c/\varepsilon)}$$

One would then expect to find c less than $\varepsilon$ because $c > \varepsilon$ would indicate that w decreased more quickly and $\theta'$ more slowly than could reasonably be expected. Thus if, as we have assumed from the outset, a and b maintain their values at all heights reached by the jet, experimental data, not necessarily very refined, can assist in the rejection of a number of theoretically possible solutions. However, the likelihood that the ratio a/b may vary from one level to another should not be excluded since a complete solution of the rising jet problem must take account of acceleration from rest in addition to what is normally the sole consideration - the deceleration from the maximum velocity attained.

Before leaving equation (8) it may be remarked that if the following condition were satisfied

$$\frac{a}{b} = \frac{4}{3(1 + c/\varepsilon)},$$

then the set of solutions would be

$$\left.\begin{aligned} w &\propto z^{-\frac{1}{3}} \\ R &\propto z \\ \theta' &\propto z^{-5/3} \end{aligned}\right\} \quad (11)$$

This set of solutions is in agreement with the results derived by Schmidt (1941), using a mixing length hypothesis, and by Rouse et al in the paper already referred to. The latters' experimental observations made in a closed room and using a gas burner as a source of heat, give the following approximate values of a and b.

$$a = 2.7; \quad b = 3.5$$

$$\text{i.e. } 3a - 2b > 0$$

With the radius of the jet increasing linearly with height, equation (8) should describe the velocity distribution. Using the value found for a and b, this requires

$$\frac{c}{\varepsilon} = \frac{7}{9}$$

Batchelor (1954) applied dimensional analysis to the problem of a continuous point source and, for turbulent flow, also obtained the set of solutions given in equations (11). His results for laminar flow are referred to in section 4.

## 3. Non-linear spreading of the jet

The object now is to investigate the solutions to equation (6) which arise when the radius of the jet increases with height but not in accordance with a

/linear

linear law. Suppose the origin of z is taken at a point where $R = R_0$ and $w = w_0$. This point $z = 0$ need not and normally would not coincide with the point source of heat. The law of variation of R with height may be written

$$R = R_c \left(\frac{z + z_0}{z_0}\right)^p \qquad (p \neq 1)$$

and equation (6) becomes

$$\frac{1}{w}\frac{dw}{dz} + \frac{2(a-b)}{(3a-2b)}\frac{h}{(z+z_0)} = -\frac{ac}{(3a-2b)}\frac{z_0^h}{R_0}(z+z_0)^{-h} \qquad (12)$$

Here we have divided through by (3a-2b) which cannot equal zero in this context since we are considering only values of p not equal to unity (see section 2(i)).

Equation (12) can be integrated and the set of solutions is given by

$$w = w_0\left(\frac{z+z_0}{z_0}\right)^{-\frac{2(a-b)}{(3a-2b)}h} \exp\left[-\frac{ac\,z_0^h}{(3a-2b)(1-p)R_0}\left\{(z+z_0)^{1-p} - z_0^{1-p}\right\}\right]$$

$$R = R_0\left(\frac{z+z_0}{z_0}\right)^p \qquad (13)$$

$$\theta' \propto \left(\frac{p}{z+z_0} - \frac{c}{R}\right) w^2$$

Here once more there is the possibility of a wide variety of solutions, depending on the values of c and p and on the ratio a/b. However, given suitable values of a, b and c and if $0 < p < 1$, the expression for the vertical velocity contained in equations (13) offers prospects of obtaining a realistic picture of velocity decay, approximating to the usual power law in the lower levels but with a more rapid rate of decay at greater heights as the exponential factor increases in importance.

Simple power law expressions for velocity decay indicate, as does the formula in (13), that the rate of ascent of the heated air approaches zero asymptotically. This is, of course, to be expected following the assumption of a neutral atmosphere but it appears that the ultimate rate of decay of the upward velocity predicted by a simple power law is too slow. Existing theories of the free convection jet may thus give disappointing results when applied, for instance, to the question of the height attained by the heated smoke plume discharged from a factory chimney. In conditions of light wind and of very small or zero gradient of potential temperature, the plume can be seen to become almost horizontal at a height substantially lower than power law theories would predict.

In the preceding section reference was made to papers in which the radius of the convection jet increases linearly with height. By contrast, Sutton (1950) drew a close analogy between the spread of a stream of hot gas and the diffusion of a cloud of cold smoke in the atmosphere and assumed that the radius of the jet, in a neutral atmosphere, would be proportioned to $z^{7/8}$, i.e. non-linear spreading. The following set of solutions was then obtained

$$w \propto z^{-7/24}$$
$$R \propto z^{7/8}$$
$$\theta' \propto z^{-35/24}$$

These solutions for w and R are contained in equations (13), except for the exponential term, provided that

$$p = \tfrac{7}{8}\ ;\ \frac{a}{b} = \frac{4}{3}$$

but the solution for $\theta'$ reduces to $\theta' \propto z^{-35/24}\left(Kz^{-1/8} - 1\right)$ where K is constant.

In the same paper, Sutton took account of the slow rate of velocity decay given by the power law. He considered the ascent of the heated gas to take place in a horizontal wind, thus relaxing the condition of an atmosphere at rest, and assumed that the decay law operated along the trajectory instead of along the vertical. A limit was set to the height attained by the plume by means of the assumption that the ascent came to an end when the trajectory was inclined at an angle of 10° to the horizontal. Even with these modifications of the power law hypothesis, the predicted rise of the plume, as has been shown by Best (1954), may greatly exceed the height observed.

The interpretation of equations (13) and their application calls for a fairly comprehensive series of observations of smoke plumes in the atmosphere in order to permit the determination of a/b, c and p for general use and of $z_o$, $w_o$, $\theta_o'$ and $R_o$ for individual cases. The writer is not aware that any programme of observations of the kind required has ever been carried out. However, the velocity formula of equations (13) can be tested in a very approximate manner by making use of some measurements of the height attained by a smoke plume at various distances downwind from the chimney stack in a steady wind of 10 feet per second. The observations were made by Bosanquet, Carey and Halton (1950) and have been used by Best (1954) in a comparison of various formulae for the assessment of ground level concentrations from a heated source.

By converting the distances downwind into times elapsed, a graph of elevation against time can be plotted and the upward velocities at various heights estimated. For application of the formula the values of p and a/b were taken as 7/8 and 4/3 respectively and, by trial, values of $z_o$, $R_o$ and c were obtained so as to give reasonable agreement with the two lowest velocities, i.e. at 0 and 10 feet above the stack. The velocities for the remaining heights were then calculated. Table I provides a comparison between the vertical velocities (1) estimated from the observations, (2) calculated by means of equations (13), and (3) calculated from the power law, i.e. with the exponential term excluded but using a separate value for $z_o$ in order to ensure agreement with the estimated velocities at 0 and 10 feet.

TABLE I

Comparison of vertical velocities estimated and calculated

Velocity (feet per second)

| Height (ft.) | Estimated from Observations | Calculated from: | |
|---|---|---|---|
| | | Equations (13) | Power Law |
| 0 | 28 | 28 | 28 |
| 10 | 14 | 13 | 14 |
| 20 | 8 | 8.4 | 11.6 |
| 30 | 5 | 6.3 | 10.3 |
| 40 | 3 | 4.8 | 9.5 |
| 50 | 1.7 | 3.9 | 8.9 |
| 60 | 1 | 3.3 | 8.5 |

It is not, of course, claimed that Table I represents an entirely valid comparison between different formulae. In the first place the atmosphere was not at rest and, in addition, it is not known whether or not the atmosphere was neutral at the time at which the observations were made. Nevertheless the comparison in Table I is encouraging and suggests that the formulae given in equations (13) may have a useful application to practical problems.

## 4. Approximate formulae for small scale convection jets

The results derived in the preceding section and given in equations (13) incorporate an exponential factor which becomes more and more important as the

vertical distance from the heat source increases. At points along the z-axis near to the source the value of

$$(z+z_o)^{1-p} - z_o^{1-p}$$

is small and especially so the smaller the value of the positive quantity (1-p).

It may seem reasonable to suppose, therefore, that for small scale experiments such as those carried out in a laboratory, the exponential factor in equations (13) will approximate to unity and may be omitted. It is to be noted that the quantity c, which expresses the effects of vertical shearing stress, appears only in the exponential part of the solutions at equations (13) and so, in practice, neglect of the exponentials, in the case of fairly narrow jets studied over a limited range of heights, is equivalent to an assumption that the effects of vertical shearing stress are small. In other words equations (13) with the exponential factors omitted are solutions of equations (3), (4) and (5) with c put equal to zero in equation (5). It is convenient to write the solutions of the basic equations, thus modified, in the following form:-

$$\left.\begin{aligned} w &= w_o \left(\frac{z+z_o}{z_o}\right)^{-\beta} \\ R &= R_o \left(\frac{z+z_o}{z_o}\right)^{\frac{1+3\beta}{2}} \\ \theta' &= \theta'_o \left(\frac{z+z_o}{z_o}\right)^{-(1+2\beta)} \end{aligned}\right\} \qquad (14)$$

In these equations $\beta = \frac{a-b}{b}$ and exact equivalence with equations (13) (excluding the exponential factors) is obtained by putting $p = \frac{3a-2b}{2b}$, a limitation on the value of p which results from neglecting the term containing c in equation (5).

Although this approximate treatment has been approached by means of the results of the preceding section with its assumption of non-linear spreading of the jet, it is possible now to relax this condition and regard equations (14) simply as solutions of the differential equation (6) with $c = 0$. It can then be seen that if $\beta = \frac{1}{3}$ is substituted in equations (14), the results given in equation (11) are reproduced, i.e. the laws of variation for w, R and $\theta'$ found by Schmidt, Rouse et al and by Batchelor.

Again if $\beta = 0$, i.e. $a = b$, a possibility already discussed in section 2(ii), equations (14) reduce to the form

$$\begin{aligned} w &= \text{constant} \\ R &\propto z^{\frac{1}{2}} \\ \theta' &\propto z^{-\frac{1}{2}} \end{aligned}$$

These are the solutions found by Batchelor for the case of laminar flow from a continuous point source of heat.

The application of the solutions given in equations (14) depends on the determination of a number of constants such as $\beta$ and $z_o$, the distance from the origin of the theoretical point source. Since the values of a and b are not known from the outset, the solutions cannot be expressed in closed form and recourse must therefore be had to experimental data. Provided these are available, a portion of the data may be used for the determination of the necessary constants and the remainder may be used to check the accuracy of the

/solutions.

solutions. Laboratory experiments under neutral conditions have been carried out by Railston (1954) and by Schmidt (1941), both of whom used small jets and whose results refer to a range of height of less than one metre. Since equations (14) were derived for small jets, the results of Railston and of Schmidt are particularly relevant and these have been used in the following sections for a quantitative test of the theory.

## 5. Comparison with Railston's experimental results

Railston used three different heat outputs - 4.87, 9.92 and 20.37 watts - and measured the temperature excess along the central axis of the rising plume at 5 cm. intervals up to a distance of 70 cm. from the source which was a square-shaped piece of fine mesh wire gauze of side 3.5 cm. In addition, for the strongest heat output, he measured the excess temperature at a number of points from the axis of the jet at the levels 20, 40 and 60 cm. above the source. This enabled graphs to be drawn showing the form of lateral temperature decay across the jet.

A point of great interest that emerged from Railston's work was that with these small jets of naturally convected air, large disturbances appeared at irregular intervals even though the jets were carefully screened and the heat supply was maintained at a constant rate. Schlieren methods were therefore used to render the air stream visible so as to ensure that measurements were made in conditions of steady flow. Railston also took steps to eliminate the effects of conduction and radiation and thus, as far as practicable, the observations referred to convection alone.

Railston made an empirical analysis of his measurements of axial excess temperature and arrived at the following relation:-

$$\theta' = h\, Q^{0.675}\, (z + 10.2)^{-1.52}$$

where Q is the heat output and h a constant. The theoretical point source from which the airstream originated was thus found to be at a position 10.2 cm. below the heated gauze.

Comparing the theoretical results derived above with Railston's empirical formula it will be seen that the power to which $(z + 10.2)$ is raised, $-1.52$, corresponds with a value of $\beta$ equal to .26. This value was verified and, using the observations for $z = 20$ cm., the appropriate form of equations (14) for $\theta'$, the temperature excess, was worked out for each heat output. The resulting formulae and a comparison with Railston's observations are set out in the following Tables I, II and III.

TABLE I

Heat output 20.37 watts

| z (cm.) | 5.0 | 10.0 | 15.0 | 20.0 | 25.0 | 30.0 | 35.0 | 40.0 | 45.0 | 50.0 | 55.0 | 60.0 | 65.0 | 70.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\theta'$ (Observed)°C | 20.5 | 12.2 | 9.2 | 7.2 | 5.4 | 4.3 | 3.8 | 3.1 | 2.6 | 2.1 | 2.0 | 1.7 | 1.4 | 1.2 |
| $\theta'$ (Calculated)°C | 20.5 | 13.3 | 9.7 | 7.2 | 5.7 | 4.6 | 3.9 | 3.3 | 2.8 | 2.6 | 2.3 | 2.0 | 1.8 | 1.7 |

TABLE II

Heat output 9.92 watts

| z (cm.) | 5.0 | 10.0 | 15.0 | 20.0 | 25.0 | 30.0 | 35.0 | 40.0 | 45.0 | 50.0 | 55.0 | 60.0 | 65.0 | 70.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\theta'$ (Observed)°C | 12.9 | 7.7 | 5.8 | 4.7 | 3.4 | 2.8 | 2.3 | 2.1 | 1.7 | 1.4 | 1.3 | 1.1 | 1.0 | 0.9 |
| $\theta'$ (Calculated)°C | 13.4 | 8.7 | 6.3 | 4.7 | 3.7 | 3.0 | 2.5 | 2.2 | 1.8 | 1.7 | 1.5 | 1.3 | 1.2 | 1.1 |



/TABLE III

TABLE III

Heat output 4.87 watts

| z (cm.) | 5.0 | 10.0 | 15.0 | 20.0 | 25.0 | 30.0 | 35.0 | 40.0 | 45.0 | 50.0 | 55.0 | 60.0 | 65.0 | 70.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\theta'$ (Observed)°C | 7.8 | 5.2 | 3.4 | 2.7 | 2.1 | 1.6 | 1.4 | 1.3 | 1.1 | 0.9 | 0.8 | 0.8 | 0.7 | 0.6 |
| $\theta'$ (Calculated)°C | 7.7 | 5.0 | 3.6 | 2.7 | 2.1 | 1.7 | 1.5 | 1.2 | 1.1 | 1.0 | 1.0 | 0.9 | 0.7 | 0.6 |

Railston's empirical formula shows $\theta'$ proportional to $Q^{0.675}$ or $Q^{2/3}$ approximately, a law predicted by Priestley and Ball as well as other writers. From the results in the above tables it can also be confirmed that the calculated $\theta'$ is roughly proportional to $Q^{2/3}$.

## 6. Comparison with Schmidt's experimental results

Schmidt used one heat output, the value of which was not given in his paper, and used a spiral coil as the source. He reported the same difficulties as Railston in obtaining a steady jet in spite of the precautions taken to exclude draughts from the laboratory.

Schmidt used a hot-wire anemometer and a small thermocouple to measure the wind and temperature in the jet and took readings at 40,50,60,70 and 80 cm. above the finite source of heat. Since this was not a true point source, he used the observations to calculate from his theoretical analysis that the equivalent imaginary point source was 11.1 cm. below the actual heat source.

Schmidt's values of wind speed and temperature excess along the axis of the jet were applied to the formulae of equations (12). The value of $\beta$ was found to be 0.29 and of $z_o$ 10.2 cm. The velocity and temperature excess along the axis were then calculated from the formulae. The results together with Schmidt's observations are shown in the following Table IV.

TABLE IV

| z cm. | $w$ Observed | $w$ Calculated | $\theta'$ Observed | $\theta'$ Calculated |
|---|---|---|---|---|
| 40 | 64 | 64 | 13.95 | 13.95 |
| 50 | 61 | 61 | 10.5 | 10.4 |
| 60 | 58 | 58 | 8.2 | 8.2 |
| 70 | 56 | 56 | 6.5 | 6.6 |
| 80 | 53 | 54 | 5.49 | 5.51 |

In this application to Schmidt's results, the value obtained for $\beta$ (0.29) gives a velocity decay law in agreement with that of Sutton's theory, but the index for temperature decay ( - 1.58) differs from Sutton's ( - 1.46).

## 7. Conclusion

The approximate solutions given in equations (14) for small jets over a limited range of height have given satisfactory agreement when applied to the experimental data presented by Railston and by Schmidt. The agreement is significantly better than that obtained by Priestley and Ball in testing their solution (section 2(i)) against Railston's results.

The data of Railston and Schmidt, combined with the theory, indicate that the values of a and b, which depend upon the lateral decay functions f(r/R) and h(r/R) are related as follows:-

(Railston) $a = 1.26\ b$
(Schmidt) $a = 1.29\ b$

/Additional

Additional laboratory data would be valuable in order both to test the formulae and to gain further insight into the magnitudes of a and b. It is considered that experiments of the type carried out by Railston but extended to the measurement of velocity as well as of temperature excess would be well worth while.

With regard to the more general formulae, equations (13), these could be tested quantitatively if comprehensive observations were made, in the appropriate atmospheric conditions, of the ascent of heated gas discharged from a chimney. The initial conditions - temperature excess at the orifice, discharge velocity, etc. - would need to be accurately measured.

References:-


| | | |
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APPENDIX

## Application to ascent of plume when atmosphere is in horizontal motion

The theoretical discussion has been concerned with ascent in a neutral atmosphere at rest, but since the ultimate objective is to obtain formulae that would be useful in the practical problems associated with chimney stacks, it is desirable to investigate the possible application of the results to a plume ascending through an environment in horizontal motion. Anything in the nature of a thorough test is at present out of the question because observations in the necessary detail are not available. However, a Report entitled "A Meteorological Survey of the Oak Ridge Area" (1953) describes photographic measurements of the height at which the plumes from different chimney stacks became horizontal. The observations were made in atmospheric conditions of small or zero temperature gradient and in various wind velocities. The Report gives a reasonable number of observations relating to two of the stacks and the data have been used to test whether the velocity formula of equations (13) could be satisfactorily applied.

The following are the design details of the two chimneys:-

| | Watts Bar Steam Plant | X-10 Steam Plant |
|---|---|---|
| Stack Height | 160 ft. | 180 ft. |
| Orifice Diameter | 14 ft. | 9 ft. |
| Exit Velocity | 34 m.p.h. | 5 m.p.h. |
| Exit Temperature | 350°F | 400°F |
| Volume Emission Rate | 300,000 cu.ft./min. | 27,200 cu.ft./min. |
| Heat Emission Rate | $6.6 \times 10^6$ cal./sec. | $7.1 \times 10^5$ cal./sec. |

Since the observational data on the plumes consisted solely of the height at which they became horizontal, it was necessary to consider carefully the values to be given to the undetermined quantities in the velocity formula.

For the spreading index, p, it seemed best to use the value 0.875 found by Sutton from numerous observations of the diffusion of smoke clouds. The quantities a and b are important only in their ratio to each other and $\frac{a}{b} = \frac{4}{3}$ was chosen following the discussion in sections 3 and 4 where it is implied that this value is appropriate to fully turbulent motion.

The Oak Ridge data gave little definite indication of the values of the constants $z_o$ and $R_o$. The radii of the chimneys are given in the Report but it cannot be assumed that these are the appropriate values to use for $R_o$. The experimental results of Railston and of Schmidt were confined to a small range of height but appear to offer the only guidance available and suggest that the value of $z_o/R_o$ is about 4. For jets on a larger scale the ratio may be greater but probably by not more than a factor of 2.

The quantity c expresses the effects of vertical shearing stress and appears only in the exponential part of the velocity formula. In sections 5 and 6 it was shown that satisfactory comparisons with experimental data could be obtained for laboratory experiments in which the range of height is necessarily small and in which reasonable precautions may be taken to ensure that the ambient air is still. It appeared, therefore, that the value of c might be related to the horizontal velocity of the environment and the test of the velocity formula was carried out on the assumption that c would increase as the external wind speed increased. Reference to the formula shows that the effect of increasing c is to increase the rate of velocity decay and this assumption is therefore supported by experience since it is well known that chimney smoke rises higher in a light wind than in a strong wind.

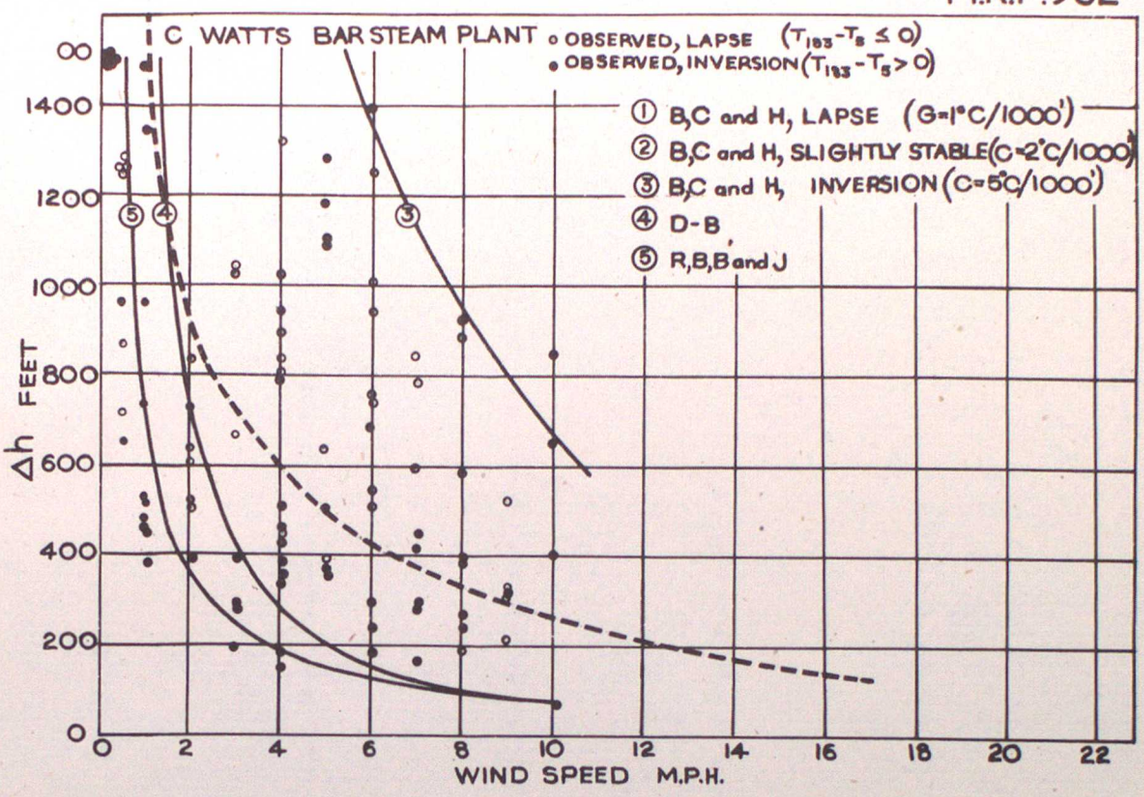
Another question was to decide the criterion for the termination of ascent. The formula for velocity indicates that the rate of ascent approaches zero

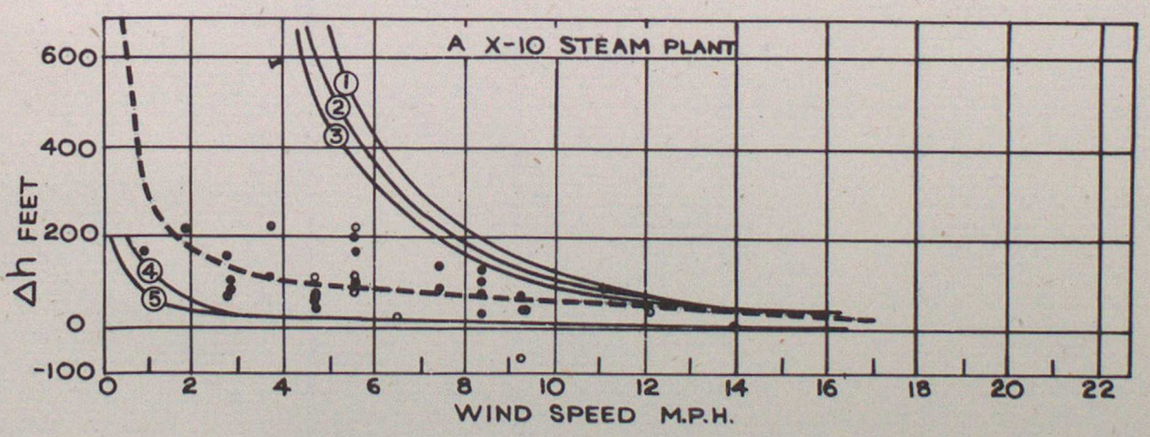
/asymptotically

asymptotically but in practice one is clearly justified in assessing a limit beyond which the residual velocity of ascent may be ignored. It was therefore decided to assume that ascent comes to an end when the upward velocity of the jet has fallen to a value equal to 10 per cent of the horizontal velocity of the environment. This condition is an exacting one and may be considered to lean to the safe side.

The attached figures showing observed plume heights plotted against wind speed are copied from the Oak Ridge Report except that the curve shown by a broken line in each graph is the result of applying the velocity formula of equations (13) adapted in the manner suggested in the preceding paragraphs. Curves in black show tests made at Oak Ridge of other theoretical formulae. Curves (1), (2) and (3) are derived from the work of Bosanquet, Carey and Halton (1950); curves (4) and (5) are based on formulae deduced from wind tunnel experiments.

The figures show that the velocity formula is capable of predicting fairly successfully the heights attained by the plumes from two very different chimneys in conditions of small temperature gradient and in a horizontal wind. However, comparisons with data from other chimneys and, when possible, with comprehensive data (observed velocity decay, increase of radius, etc.) from at least one chimney are required before the formula could be applied generally. Comparisons on these lines will be undertaken as soon as data become available.




OBSERVED AND CALCULATED PLUME HEIGHTS